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NOTES ON MURPHY'S METHOD

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ABSTRACT. A mathematical justification is provided for some of the simpler versions of Murphy's method in nonlinear ballistics.

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FOREWORD

This report was originally written to serve (and be distributed) as lecture notes for a talk given by the author as guest lecturer for a course in "The Free Flight Motion of Symmetrical Missiles," which was given by Dr. C. H. Murphy. This course and his talk, an intensive one-week one, has been given twice (fall of 1961 and 1962) at the University of California at Los Angeles, and once (fall 1962) when sponsored by the Naval Weapons Laboratory. The report is being published to make it more readily available to interested parties.

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NOTES ON MURPHY'S METHOD

Given the equation from Murphy, Ref. 1:

$$\ddot{z} + (H-\dot{\ell}/\ell-i\omega)\,\dot{z} - (M+i\omega T)\,z = 0 \tag{1}$$

where $\ell = \sqrt{1-|z|^2}$, and H, M, and T are real, may be functions of $|z|^2$, and ω is real, we wish to draw rigorously justified conclusions about the nature of its family of solutions.

In the first place, the origin $z = \dot{z} = 0$ is a singular point, a solution. The conditions for stability are well known:

$$\begin{aligned} &H > 0 \\ &(\omega^2-4M)\,H^2 > \omega^2\,(2T-H)^2 \end{aligned} \tag{2}$$

with H, M, T evaluated at $z = 0$.

In the second place, it turns out that under certain conditions there is one (or more) solution of the form $z = re^{i\nu t}$, with r and ν constant, a <u>periodic</u> solution. We postpone discussion for a bit.

In the third place we are interested in the following situation, if it should occur:

There is a set of four real, smooth functions $X_i(\theta_1,\theta_2)$ each periodic in θ_1 and θ_2 with periods $1/\omega_1$ and $1/\omega_2$, respectively, and for each choice of θ_{10}, θ_{20} there exist two real, differentiable functions $\theta_1(t)$ and $\theta_2(t)$ such that $\theta_i(0) = \theta_{i0}$ and with $z = z_1 + i\,z_2$, $x_1 = z_1$, $x_2 = \dot{z}_1$, $x_3 = z_2$, $x_4 = \dot{z}_2$, $x_i(t) = X_i(\theta_1(t),\,\theta_2(t))$ is a solution of Eq. 1 and the only one with $x_i(0) = X_1(\theta_{10},\theta_{20})$. If this should be true, we say that the vector function $X_i(\theta_1,\theta_2)$ defines a periodic surface of Eq. 1.

We know no way to establish the existence of such surfaces for Eq. 1 as it stands. If, however, we are willing to regard the nonlinearities and dissipative terms as small, definite results can be established. We therefore work with

$$\ddot{z} - i\omega\dot{z} - Mz + \epsilon[(H-\dot{\ell}/\ell)\,\dot{z} - (m+i\omega T)\,z] = 0 \tag{3}$$

where M = constant, and H, m, T depend on $|z|^2$ only, m(0) = 0. We shall use this form in hunting for periodic solutions (conical yaw) as well as periodic surfaces (steady mixed oscillations). Let $z = \rho e^{i\varphi}$, then Eq. 3 is equivalent to the pair:

$$\begin{aligned} &\ddot{\rho} - \rho\dot{\varphi}^2 + \omega\rho\dot{\varphi} - M\rho + \epsilon[(H+\rho\dot{\rho}/(1-\rho^2))\,\dot{\rho} - m\rho] = 0 \\ &\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} - \omega\dot{\rho} + \epsilon[H\rho\dot{\varphi}-\omega T\rho] + \epsilon\rho^2\dot{\rho}\dot{\varphi}/(1-\rho^2) = 0 \end{aligned} \tag{4}$$

This set does <u>not</u> depend on φ itself. We look for a pair $\rho = \rho_0$, $\dot{\varphi} = \nu$ such that according to Eq. 4 (with $\dot{\rho} = 0$), $\ddot{\rho} = \ddot{\varphi} = 0$. The first is satisfied if

$$\nu^2 - \omega\nu + (M+\epsilon m(\rho_0^2)) = 0 \tag{5}$$

or

$$\nu = \frac{1}{2}\left(\omega \pm \sqrt{\omega^2 - 4(M+\epsilon m(\rho_0^2))}\right)$$

(provided, of course, that this is real).

The second is satisfied if, simultaneously

$$H\nu - \omega T = 0 \tag{6}$$

To investigate the behavior in the neighborhood of such a periodic solution, we shall, temporarily set $\rho = \rho_0 + x_1$, $\dot{\rho} = x_2$, $\dot{\varphi} = \nu + x_3$ and expand system 4.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \begin{cases} 2\rho_0 \nu x_3 - \omega\rho_0 x_3 \\ + \epsilon \begin{bmatrix} -Hx_2 - mx_1 - m'\rho_0 x_1 \\ -\dfrac{2\rho_0 \nu}{2\Omega} m' x_1 \end{bmatrix} \\ \qquad + 0(x^2) \end{cases} \tag{7}$$

$$\rho_0 \dot{x}_3 = \begin{cases} -2\nu x_2 + \omega x_2 \\ + \epsilon[-H\rho_0 x_3 - H'\rho_0 \nu x_1 + \omega\rho_0 T' x_1] \\ + 0(x^2) + \epsilon\rho_0^2 \nu x_2/(1-\rho_0^2) \end{cases}$$

with $2\Omega = \pm\sqrt{\omega^2 - 4(M+\epsilon m)} = 2\nu - \omega$. It is well known that for a system such as

$$\dot{x} = Ax + 0(x^2)$$

where x is a vector and A a constant matrix, that the behavior of the solutions near x = 0, including stability criteria, is characterized by the roots of the 'secular equation' det $|A-\lambda E| = 0$, at least when det $|A| \neq 0$. For simplicity, and to be able to compare directly with pages 26 and 27 of Ref. 1, we will specialize to H = constant, m = 0. The secular equation is then (we also neglect $\epsilon\rho_0^2 \nu/(1-\rho_0^2)$)

$$\lambda^3 + 2\epsilon H\lambda^2 + [4\Omega^2 + \epsilon^2 H^2]\lambda - 2\epsilon\omega\Omega\rho_0 T' = 0 \tag{8}$$

The periodic solution is stable if and only if the real parts of all three roots of Eq. 8 are negative, which in turn is true if and only if (neglecting the ϵ^2 term),

$$H > 0 \ , \ \omega\,\Omega\rho_0 T' < 0$$
$$H + \omega\rho_0 T'/4\Omega > 0 \tag{9}$$

(of which the first happens to be a consequence of the second and third). We can even find approximate values for the roots

$$\lambda_1 \cong \epsilon\omega\rho_0 T'/2\Omega$$

$$\lambda_{2,3} \cong \pm\ 2i\Omega - \epsilon(H+\omega\rho_0 T'/4\Omega)$$

Equations 65 and 66 of Ref. 1 lead to only two roots:

$$\lambda_1 = \epsilon\omega\rho_0 T'/2\Omega$$
$$\lambda_2 = -\ \epsilon(H+\omega\rho_0 T'/4\Omega)$$

Now let $\nu_1 = \omega/2 + \Omega$, $\nu_2 = \omega/2 - \Omega$, $4\Omega^2 = \omega^2 - 4M$, and suppose $\Omega \neq 0$. Make the transformation

$$z = r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2}$$

$$\dot{z} = ir_1\nu_1 e^{i\varphi_1} + ir_2\nu_2 e^{i\varphi_2}$$

with the understanding that neither r_1 nor r_2 is zero. Just as in Ref. 1, we obtain, setting

$$\theta_1 = \varphi_1 + \varphi_2\ ,\ \theta_2 = \varphi_1 - \varphi_2$$

$$\begin{aligned}
\dot{\theta}_1 &= \nu_1 + \nu_2 + \epsilon\ \Theta_1\ (r_1,r_2,\theta_2) + O(\epsilon^2)\\
\dot{\theta}_2 &= \nu_1 - \nu_2 + \epsilon\ \Theta_2\ (r_1,r_2,\theta_2) + O(\epsilon^2)\\
\dot{r}_1 &= \frac{\epsilon}{\nu_1-\nu_2} R_1(r_1,r_2,\theta_2) + O(\epsilon^2)\\
\dot{r}_2 &= \frac{\epsilon}{\nu_1-\nu_2} R_2(r_1,r_2,\theta_2) + O(\epsilon^2)\\
R_1 &= \begin{cases} (-H\nu_1+\omega T)r_1 + \begin{bmatrix}(-H\nu_2+\omega T)\cos\theta_2\\ -\ m\ \sin\ \theta_2\end{bmatrix} r_2\\ +\ \dfrac{r_1r_2(\nu_1-\nu_2)\sin\theta_2(r_1\nu_1+r_2\nu_2\cos\theta_2)}{1-\delta^2}\end{cases}\\
R_2 &= \begin{cases} (H\nu_2-\omega T)r_2 + \begin{bmatrix}(H\nu_1-\omega T)\cos\theta_2\\ +\ m\ \sin\ \theta_2\end{bmatrix} r_1\\ -\ \dfrac{r_1r_2(\nu_1-\nu_2)\sin\theta_2(r_2\nu_2+r_1\nu_1\cos\theta_2)}{1-\delta^2}\end{cases}
\end{aligned} \tag{10}$$

with $\delta^2 = r_1^2 + r_2^2 + 2r_1r_2 \cos\theta_2$, m, H and T functions of δ^2. The right-hand sides of all four equations are independent of θ_1, so we may consider the last three alone. Θ_i and R_i are of period 2π in θ_2. Let

$$H_1 = \frac{1}{2\pi}\int_0^{2\pi} H \, d\theta$$

$$H_2 = \frac{1}{2\pi}\int_0^{2\pi} H \cos\theta \, d\theta$$

$$T_1 = \frac{1}{2\pi}\int_0^{2\pi} T \, d\theta$$

$$T_2 = \frac{1}{2\pi}\int_0^{2\pi} T \cos\theta \, d\theta$$

We find that

$$\overline{R_1} = \frac{1}{2\pi}\int_0^{2\pi} R_1 \, d\theta = -(H_1\nu_1-\omega T_1)\, r_1 - (H_2\nu_2-\omega T_2)\, r_2$$

$$\overline{R_2} = \frac{1}{2\pi}\int_0^{2\pi} R_2 \, d\theta = (H_1\nu_2-\omega T_1)\, r_2 + (H_2\nu_1-\omega T_2)\, r_1$$

Suppose we can find a pair r_{10}, r_{20} such that both $\overline{R_1}$ and $\overline{R_2}$ are zero. We can then apply a theorem of Poincaré (see theorem 5.2 of Ref. 2) to the effect that, (subject to certain differentiability conditions) if also the determinant of the matrix $\frac{\partial \overline{R}_i}{\partial r_j}$ is <u>not</u> zero, there is an $\epsilon_1 > 0$ such that for $|\epsilon| < \epsilon_1$ a solution does exist for which r_1 and r_2 are periodic functions of θ_2, and therefore of the time, θ_2 increasing (if $\nu_1 > \nu_2$) strictly monotonically.

The z_1, z_2, $\dot{z}_1$ and $\dot{z}_2$ calculated from this $r_1(\theta_2)$, $r_2(\theta_2)$ and the first of Eq. 10 define just such a periodic surface as we were looking for.

Let the matrix $\partial \overline{R_i}/\partial r_j = A$. It can be shown that the character of the family of solutions sufficiently near such a periodic surface is determined by the roots of det $|A-\lambda E| = 0$ in the usual way.

Applying these results to the case discussed on pages 28 to 30 of Ref. 1, one obtains identical conclusions.

We see that the methods of Ref. 1 are thoroughly justified, in a perturbation theory sense, insofar as they deal with the 'singular points' of the 'amplitude plane' and their neighborhoods, subject to a minor modification when the critical point is on an amplitude axis (r_1 or r_2 zero).

It is possible to extend this treatment to cover the cases of strongly nonlinear moments, but small yaw, discussed in Ref. 3. It is even possible to include all the kinematic nonlinearities (such as the $\dot{\ell}/\ell$ term) in the zeroth approximation as well. One proceeds as follows: Write the equation

$$\ddot{z} - [i\omega + \dot{\ell}/\ell]\,\dot{z} - \ell M z + \epsilon[H\dot{z} - (m + i\ell\omega T)z] = 0$$

Then set $z_1 = \sin\rho \cos\varphi$, $z_2 = \cos\rho \sin\varphi$, and finally $\dot{\varphi} = \dot{\psi}\cos\rho$. One obtains a pair of equations for $\ddot{\rho}$ and $\ddot{\psi}$, which are essentially those of Ref. 4. We can search for a steady state solution of this pair just as before. We can even take $\epsilon = 1$ and get an answer. This procedure, and the calculation of stability criteria is carried out in Ref. 4.

Now define

$$u = \cos\rho$$

$$v = -2\int_0^\rho M \sin\rho \, d\rho = 2\int_1^u M du$$

$$h = \dot{\rho}^2 + \dot{\psi}^2 \sin^2\rho + v$$

$$q = \dot{\psi}\sin^2\rho + \omega u$$

h and q are closely related to C_1 and C_2, respectively, of Ref. 3. We will find that both $\dot{h}$ and $\dot{q}$ are of order ϵ, and both are well behaved functions of $\dot{u}$, u, h and q. Furthermore

$$\dot{u}^2 = (1-u^2)(h-v) - (q-\omega u)^2 \equiv F(u,h,q)$$

If on the interval [-1,1], $F(u,h,q)$ has two simple roots separated by an open interval on which $P > 0$, then if h and q were constant, u would have a periodic solution $P(t,h,q)$ of period $1/\nu(h,q)$. Use that one for which $u(0) = u_1$ = least of the pair of roots.

Let $p(\theta,h,q) = P(\nu(h,q)\theta,h,q)$. Now set $u = p(\theta,h,q)$, $\dot{u} = \nu(h,q)\dfrac{\partial}{\partial\theta}p(\theta,h,q)$.

$\dot{h}$ and $\dot{q}$ are now functions of θ, h and q, and $\dot{\theta}$ may be calculated from

$$\nu\, p_\theta = \dot{u} = \dot{\theta}\, p_\theta + \dot{h}\, p_h + \dot{q}\, p_q$$

$$\dot{\theta} = \nu(h,q) - \frac{\dot{h}p_h + \dot{q}p_q}{p_\theta}$$

It can be shown that this expression is well behaved even at $\theta = 0$ and $1/2$ where $\dot{u} = \nu p_\theta = 0$. We have in fact obtained a system

$$\begin{aligned}\dot{\theta} &= \nu(h,q) + \epsilon\,\Theta(\theta,h,q)\\ \dot{h} &= \epsilon\, H(\theta,h,q)\\ \dot{q} &= \epsilon\, Q(\theta,h,q)\end{aligned} \qquad (11)$$

with Θ, H and Q periodic (of period 1) in θ. Theorem 5.2 of Ref. 2 may again be applied with entirely similar results.

A detailed discussion of the more general version of the problem is given in Ref. 5. Because it is possible to reduce the problem to third order (the last three equations of 10 or the set 11), the theory of periodic surfaces has not, in fact been needed, but only that of periodic solutions. On the other hand, if the right side of Eq. 11 or 3 is a constant $G \neq 0$, the full theory of periodic surfaces is required to deal with the question of steady mixed modes. A discussion of some simple cases with $G \neq 0$ is contained in Ref. 6.

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